THE CORIOLIS EFFECT – WHAT’S GOING ON?

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Abstract – *The subject of dynamics is fundamental to engineering programs in general, and can be challenging to teach. In particular, the topic of Coriolis acceleration and the “Coriolis effect” may present the greatest challenge of all. Although mathematical formulations and physical explanations are available in many textbooks and technical articles, undergraduate students still have trouble making meaningful and long-lasting connections between the mathematical expressions and the physical manifestations of the phenomenon. In our opinion, a significant part of the problem can be attributed to the traditional formulation of kinematic equations using rotating frames of reference. Normally, the Coriolis acceleration is defined, derived, and explained using at least one noninertial frame. At the University of Saskatchewan, our approach is to express the equations of rigid body dynamics solely with respect to a single inertial frame of reference, and to thereby solve all the problems that would otherwise be solved using rotating frames. We also use this approach for deriving and explaining the Coriolis effect. Although our approach is, in a sense, equivalent to the traditional method, we have found that students are better able to understand and apply the principles of dynamics when working within a single inertial frame. In this paper, our pedagogical approach will be presented and then applied to the question of “what’s going on” with the Coriolis effect.*

Keywords: Coriolis Effect; Rigid Body Dynamics; Kinematics; Inertial Frame of Reference

1. INTRODUCTION

Any course in engineering mechanics should provide a set of fundamental principles and governing equations applicable to a wide variety of problems. The formulation should be clear and consistent, thus empowering the student to meet any technical challenge. In undergraduate courses on rigid body dynamics, the most difficult problems typically involve multi-part systems with 3-D motion, together with sliders or pins moving along rotating parts. In most undergraduate

textbooks, this leads to the introduction of rotating frames of reference, and relative acceleration components written in terms of the rotating frames. One of these components is referred to as the “Coriolis acceleration”. In the textbook by R.C. Hibbeler [1], for example, the Coriolis acceleration is defined as “the difference in the acceleration of [a point] B as measured from nonrotating and rotating x , y , and z axes”. Other textbooks for undergraduate dynamics contain similar definitions; e.g. [2-5]. This method of analysis then requires judicious selection of multiple frames, each with its own angular velocity at a given instant.

In our experience, this approach is suboptimal in promoting an understanding of the physical nature of the motion, as well as its connection to the mathematical formulation. In addition, considerable confusion can result when comparisons are made between textbooks. Whereas some books use the rotating frames as an intermediate tool to determine the “absolute accelerations” within an inertial frame of reference, others focus on the effects observed from a rotating frame, such as the Earth. Since Newton’s laws do not apply in a noninertial frame of reference, the observed “Coriolis effects” and “Centrifugal effects” are accompanied by corresponding, compensating fictitious forces that are only manifestations of the rotating frame.

In our pedagogical practice, we advocate the use of a single inertial frame of reference for all analyses. The equations of relative position, velocity, and acceleration are all derived for this single inertial frame, and the kinetic analysis includes only those forces due to natural fields (such as gravity) and direct contact between objects. Our approach does not require the selection of rotating frames, and kinematic analyses are performed systematically by applying the same fundamental equations repeatedly between joints and connecting points in a mechanism, even for 3-D problems. Ironically, once dynamics in an inertial frame is clearly understood, this has provided an effective starting point for explaining the effects in rotating frames described by G.G. Coriolis [6]. In the next section, we develop our kinematic approach as it would be presented to a class of second or third year undergraduate students. Special attention is paid to “Coriolis-related” terms.

2. KINEMATIC FORMULATION

Our approach begins with definition of the relative position vector $\vec{r}_{B/A}$ between two points in space, as illustrated in Fig. 1. In practice, A and B would be two points on a mechanism, and, by definition, the unit vector $\hat{u}_{B/A}$ is always along $\vec{r}_{B/A}$.

Fig. 1. Relative Position Vector.

We emphasize that $\vec{r}_{B/A}$, being a vector quantity, can change in magnitude or direction or both, and that these changes can be observed from an inertial frame of reference which we treat as fixed relative to the stars. We further emphasize that this simple but fundamental fact is true for all the vector quantities to be encountered.

The time rate of change due to a change in magnitude is given by

$$\left(\frac{d\vec{r}_{B/A}}{dt} \right)_{\text{mag}} = \frac{dr_{B/A}}{dt} \hat{u}_{B/A}, \quad (1)$$

where t is time and $r_{B/A}$ (without the arrow) is the magnitude of $\vec{r}_{B/A}$. At any instant, this change occurs along the vector $\vec{r}_{B/A}$, and therefore we refer to it as the “stretch velocity” of B with respect to A.

Deriving the time rate of change due to a change in direction requires more work, but provides the basis for physical interpretations later in the analysis. A physical description is provided by considering a change in the direction of $\vec{r}_{B/A}$ over a small time increment dt , as shown in Fig. 2 for 2-D, but easily extended to 3-D.

Fig. 2. Rotation of Relative Position Vector.

By the definition of vector addition, the change in the relative position vector is given by vector $d\vec{r}_{B/A}$ between the respective tips of $\vec{r}_{B/A}$ at times t and $t + dt$. In the limit, the magnitude of the time derivative “due to rotation” becomes

$$\left| \frac{d\vec{r}_{B/A}}{dt} \right|_{\text{rot}} = \frac{r_{B/A} d\theta}{dt} = r_{B/A} \omega, \quad (2)$$

where ω is the magnitude of the angular velocity $\vec{\omega}$ of the position vector. Defining the direction of $\vec{\omega}$ as perpendicular to the plane of rotation, with its sense defined by the right hand rule, we obtain

$$\left(\frac{d\vec{r}_{B/A}}{dt} \right)_{\text{rot}} = \vec{\omega} \times \vec{r}_{B/A}, \quad (3)$$

where the cross product gives the correct direction of $d\vec{r}_{B/A}$, perpendicular to both $\vec{\omega}$ and $\vec{r}_{B/A}$. Here, we emphasize that the time rate of change in any vector due to its change in direction can be expressed as the cross product of its angular velocity with the vector itself, as in eqn. (3). In other words, we can think of the angular velocity as “turning” the vector (by means of the cross product) to effect its change. Finally, noting that the time derivative of relative position gives the relative velocity $\vec{v}_{B/A}$, equations (1) and (3) are combined to give

$$\vec{v}_{B/A} = \frac{dr_{B/A}}{dt} \hat{u}_{B/A} + \vec{\omega} \times \vec{r}_{B/A}. \quad (4)$$

Equation (4) is a general relation for any two points in space, and can be used for both 2-D and 3-D analyses without the introduction of rotating frames of observation.

To derive the relative acceleration equation, we simply differentiate eqn. (4), with all derivatives performed with respect to the inertial frame of reference. Using the product rule of differentiation, we obtain

$$\begin{aligned} \vec{a}_{B/A} &= \frac{d\vec{v}_{B/A}}{dt} \\ &= \frac{d^2 r_{B/A}}{dt^2} \hat{u}_{B/A} + \frac{dr_{B/A}}{dt} \frac{d\hat{u}_{B/A}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt}, \end{aligned} \quad (5)$$

or, if we define the time derivative of $\vec{\omega}$ as angular acceleration $\vec{\alpha}$, eqn. (5) becomes

$$\vec{a}_{B/A} = \frac{d^2 r_{B/A}}{dt^2} \hat{u}_{B/A} + \frac{dr_{B/A}}{dt} \frac{d\hat{u}_{B/A}}{dt} + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt}. \quad (6)$$

Now, since $\hat{u}_{B/A}$ is a unit vector and therefore constant in magnitude, it can change only in direction. Thus, invoking the principle established with eqn. (3), the second term of eqn. (6) becomes

$$\frac{dr_{B/A}}{dt} \frac{d\hat{u}_{B/A}}{dt} = \frac{dr_{B/A}}{dt} \vec{\omega} \times \hat{u}_{B/A} \quad (7)$$

or, rearranging,

$$\frac{dr_{B/A}}{dt} \frac{d\hat{u}_{B/A}}{dt} = \vec{\omega} \times \frac{dr_{B/A}}{dt} \hat{u}_{B/A}. \quad (8)$$

From close examination of the steps leading to the second term of eqn. (6) and the result given by eqn. (8), we conclude that part of the acceleration $\vec{a}_{B/A}$ is due to a change in direction of the stretch velocity.

In addition, the final term in eqn. (6), resulting from a change in $\vec{r}_{B/A}$, can be rewritten by noting that the derivative of $\vec{r}_{B/A}$ is, in fact, the relative velocity given by eqn. (4); that is, the final term of eqn. (6) is given by

$$\vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt} = \vec{\omega} \times \left(\frac{dr_{B/A}}{dt} \hat{u}_{B/A} + \vec{\omega} \times \vec{r}_{B/A} \right) \quad (9)$$

or, again rearranging,

$$\vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt} = \vec{\omega} \times \frac{dr_{B/A}}{dt} \hat{u}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \quad (10)$$

We note that yet another term, mathematically identical to the right hand side of eqn. (8) has emerged, but from a completely different physical cause. In this case, the first term on the right side of equation (10) exists due to the change in $r_{B/A}$, thus changing the $\vec{\omega} \times \vec{r}_{B/A}$ velocity component in magnitude.

Substituting eqn. (8) and eqn. (10) into eqn. (6) gives the final equation to be used for relative acceleration analysis, namely

$$\vec{a}_{B/A} = \frac{d^2 r_{B/A}}{dt^2} \hat{u}_{B/A} + 2 \vec{\omega} \times \frac{dr_{B/A}}{dt} \hat{u}_{B/A} + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \quad (11)$$

Each of the terms in equation (11) has been derived with reference to an inertial frame, and each has a direct

physical interpretation. The first term is the stretch acceleration due to a change in magnitude of the stretch velocity. The third term, $\vec{\alpha} \times \vec{r}_{B/A}$, expresses the change in velocity component $\vec{\omega} \times \vec{r}_{B/A}$ due to a change in $\vec{\omega}$. Again invoking the principle established with equation (3), the fourth term, $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$, represents the change in $\vec{\omega} \times \vec{r}_{B/A}$ due to a change in its direction. The vector $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$ is therefore perpendicular to $\vec{\omega} \times \vec{r}_{B/A}$, and is the relative centripetal acceleration of point B with respect to A.

Most notable, however, is the way in which the process has revealed the physical meaning of the second term in eqn. (11); i.e.,

$$(\vec{a}_{B/A})_{\text{coriolis}} = 2 \vec{\omega} \times \frac{dr_{B/A}}{dt} \hat{u}_{B/A}. \quad (12)$$

Although we have observed all vectors and vector changes from an inertial frame of reference, we have obtained an expression that gives identically the component referred to as “Coriolis acceleration” in most textbooks in engineering mechanics. The reason for the factor of “2” is plain; there are, in fact, two separate physical phenomena which are *mathematically* equivalent. Although eqn. (12) is concise in form, perhaps it would be more instructive, based on the foregoing derivation, to re-write it as two separate terms, thus:

$$(\vec{a}_{B/A})_{\text{coriolis}} = \vec{\omega} \times \left(\frac{dr_{B/A}}{dt} \hat{u}_{B/A} \right) + \left(\vec{\omega} \times \frac{dr_{B/A}}{dt} \right) \hat{u}_{B/A}. \quad (13)$$

Interestingly, each of the terms in equation (13) is due to an angular velocity combined with a stretch velocity, without the need for an angular acceleration or an increase in the rate of “stretch”. In the form given by eqn. (13), the “cross-talk” between rotation and stretching is emphasized. The first term is seen to result from the change in direction of the stretch velocity due to rotation, as per the principle of eqn. (3). The grouping in the second term emphasizes the change in the magnitude of rotation-based velocity component $\vec{\omega} \times \vec{r}_{B/A}$ due to stretch rate $dr_{B/A}/dt$. (In emphasizing the combination of rotation and stretch velocity, this definition is similar to that normally given in mechanics texts, except that, in the latter, the rotation refers to a rotating frame of reference, and the stretch velocity is described as a relative velocity as observed from that rotating frame.)

To further illustrate the physical meaning of eqn. (13), we consider a rod-slider assembly, where the rod and slider are rotating with a constant angular velocity $\vec{\omega}$,

and the slider is moving along the rod with a constant stretch speed $dr_{B/A}/dt$. The first term in eqn. (13) is depicted by Fig. 3, while the second term is depicted in Fig. 4. In both figures, the subscript B/A has been omitted in all velocity terms to avoid clutter.

Fig. 3. Change in Stretch Velocity Component Due to Rotation.

Fig. 4. Change in Rotation-Based Velocity Component Due to Stretch.

In Fig. 3, the difference between the stretch velocity at time $t + dt$ and the stretch velocity at time t gives a resultant vector between the tips of the stretch velocity vectors which, in the limit, is perpendicular to the relative position vector $\vec{r}_{B/A}$. In Fig. 4, the difference between vectors $\vec{\omega} \times r' \hat{u}$ and $\vec{\omega} \times r \hat{u}$ gives a net result in the same direction as $\vec{\omega} \times r \hat{u}$, and therefore again perpendicular to the relative position vector $\vec{r}_{B/A}$. Together, eqn. (13), Fig. 3, and Fig. 4, may be used to explain the physical basis of the Coriolis acceleration calculated as a part of kinematic analyses in rigid body dynamics. Students should apply relative velocity equation (4) and relative

acceleration equation (11) to a large variety of problems to gain confidence and establish a solid foundation of understanding. After this foundation is established, they are in a position to consider one last question – how does the physical interpretation developed in the context of an inertial frame connect with the “effects” commonly presented for rotating frames? This question is addressed in the next section.

3. THE CORIOLIS “EFFECT”

Although we strongly advocate the use of an inertial frame of reference for analysis and physical interpretation in rigid body dynamics, we also recognize that rotating frames are widely used by others. G.G. Coriolis, himself, focused on the additional “forces” that must be added to permit a kinetic analysis by an observer in a rotating frame [7]. The field of meteorology [8] commonly uses a frame of reference rotating with the Earth, and the term “Coriolis force” can be found in many engineering sources. In fact, many would argue that the term “Coriolis” should not be used except for effects observed in a rotating frame. We encourage our students to use the inertial frame approach since, in our experience, it has greatly empowered them to solve complex problems with understanding and confidence. At the same time, it is possible to provide a conceptual bridge that will facilitate discussion and interactions with those who use rotating frames. This connection is provided by the concept of relative motion. One possible explanation for students is as follows.

The two phenomena represented in Figs. 3 and 4 were presented with all motion observed from a fixed frame in space. The resulting Coriolis acceleration relied on two velocity components: a stretch velocity, $(dr_{B/A}/dt)\hat{u}_{B/A}$, and the velocity component related to rotation, namely $\vec{\omega} \times \vec{r}_{B/A}$. Both the stretching motion and the rotational motion were “real”, relative to a fixed frame in space, and the resulting accelerations would necessarily be the result of real forces in nature. If the rod AB had been fixed with the slider B still moving along the rod with a stretch velocity, there would be no Coriolis acceleration with respect to the fixed frame. If, however, the fixed rod with translating slider were observed by someone in a rotating frame of reference with angular velocity $\vec{\Omega}$ (and the observer did not perceive that he or she was rotating), the observer would think that it is the rod which is rotating with angular velocity in the opposite direction; i.e., $-\vec{\Omega}$. A translational analogy is the case where an observer is in a car accelerating past a tree on the side of the road. If the accelerating observer thinks of the car as fixed, then the observer will conclude that it must be the tree which is accelerating backwards. Moreover, from the point of view of this observer, there must be a force *causing* the

tree to accelerate. In the same way, a Coriolis acceleration is observed, and a Coriolis force implied, when a particle translating along a straight line is viewed from a rotating frame.

We find that this translational analogy effectively emphasizes (to the student) the nature of “Coriolis effects” resulting solely from a rotating point of view. The analogy is fairly direct, in that acceleration can originate kinematically from two sources: changes in velocity due to changes in magnitude (as in the car), and changes in velocity due to rotation. Similarly, a frame can be noninertial due to translational acceleration, and/or rotation.

Once the concept of relative motion in terms of frame rotation is understood, the student can better appreciate the Coriolis effects traditionally described by others, in terms of $-2\vec{\Omega} \times \vec{v}_{\text{rel}}$, where $\vec{v}_{\text{rel}}$ is the velocity viewed from the rotating frame.

4. CONCLUDING REMARKS

Elements of our approach can be found in various sources. For example, Tongue and Sheppard [9] give a similar physical explanation of Coriolis acceleration in the context of 2-D polar coordinates. They also connect the cylindrical components of acceleration in particle dynamics to the corresponding kinematic quantities for a rigid body. Boresi [10] and Bedford and Fowler [11] define the Coriolis acceleration using polar coordinates in 2-D, and then rotating frames in 3-D. Meriam and Kraige [3], as well as Boresi [10], present the derivatives of unit vectors in terms of cross products with angular velocity. Other such examples can be found in undergraduate texts. To the best of the authors’ knowledge, however, the specific 3-D formulation presented here, with its emphasis on a single inertial frame of reference, as well as the stretch and rotation of relative position vectors with respect to that inertial frame, is not currently available in textbooks in rigid body dynamics. In particular, essentially all the available texts incorporate rotating frames at some point of the presentation, whether it be in the derivations, the governing equations, or as part of the solution technique. In this article, we have shown that deriving the relative acceleration equation from an inertial frame naturally leads to the physical interplay between stretch and rotation of the relative position vector, thus uncovering the two parts of Coriolis acceleration: change in the direction of stretch as a result of rotation, and change in rotation-based velocity as a result of stretch.

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