Optimal Design of Two k -out-of- n Systems Connected in Series

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Abstract

The safety monitoring system in an underground mine consists of mainly methane sensors and smoke sensors. It can be described as a k -out-of- n :F system with dual failure modes. This paper is to determine the values of k_1^ and k_2^* that maximize system reliability and minimize average system cost for fixed n_1 and n_2 values; and the values of n_1^* and n_2^* that maximize system reliability and minimize average system cost for fixed k_1 and k_2 values. Two approaches to find n_1^* and n_2^* for fixed k_1 and k_2 values by explicitly considering the constraint $n_1^* + n_2^* \leq n$ will also be illustrated. The results presented in this paper are useful for optimal design of safety monitoring systems in underground mines.*

1. Introduction

Reliability practices began in the late 1940's. With rapid development of technologies in electronics, components and systems have become increasingly complex and reliability has been given more considerable attention by both researchers and practitioners. From the viewpoint of engineering, reliability engineering deals with measures to extend a component or system's life by reducing its potential failure modes.

Many systems use redundancy to achieve their intended reliability. A common form of redundancy is a k -out-of- n :F system in which at least k out of n components must be failed for the system to fail. In practice, some systems consist of two subsystems which have a k -out-of- n :F structure individually. One example is the monitoring sensor group and controller subsystem applied in underground coal mines. It should be noted that existing results for the single k -out-of- n :F system cannot directly provide an optimal solution for this kind of subsystem structure. In this paper, a new model for this problem will be presented, which is called *two k -out-of- n :F subsystems connected in series*.

2. Review of k -out-of- n Systems

A common form of redundancy in fault-tolerant systems is known as the k -out-of- n system, introduced by Birnbaum et al. [2] in 1961. In such a k -out-of- n system, both the system and its components have only two states: working (good) or failed. The term k -out-of- n system usually indicates a k -out-of- n :G system, a k -out-of- n :F system, or both.

Many types of systems consist of components that can fail in two mutually exclusive modes; correspondingly, the system can fail in either of two mutually exclusive ways. We call it a k -out-of- n :F system with two failure modes. The probability of a system failing in closed mode is denoted as $F_s(k, n)$ [6]:

$$F_s(k, n) = \sum_{i=k}^n \binom{n}{i} q_s^i p_s^{n-i} = 1 - \sum_{i=0}^{k-1} \binom{n}{i} q_s^i p_s^{n-i}, \quad (1)$$

and the probability of a system failing in open mode is denoted as $F_o(k, n)$ [6]:

$$F_o(k, n) = \sum_{i=n-k+1}^n \binom{n}{i} q_o^i p_o^{n-i} = \sum_{i=0}^{k-1} \binom{n}{i} p_o^i q_o^{n-i}, \quad (2)$$

where, q_s is the probability of each component failing in closed mode ($p_s = 1 - q_s$), and q_o is the probability of each component failing in open mode ($p_o = 1 - q_o$) when the components are all independent and identically distributed (i.i.d.).

System reliability is denoted as $R(k, n)$:

$$R(k, n) = 1 - F_s(k, n) - F_o(k, n), \quad (3)$$

and the average system cost is denoted as $T(k, n)$:

$$T(k, n) = c_0 n + c_1 F_s(k, n) + c_2 F_o(k, n), \quad (4)$$

where, c_0 is the cost of each component, c_1 is the cost when system failure is in closed mode, and c_2 is the cost when system failure is in open mode.

Studies of k -out-of- n systems with two failure modes began in the 1960's [4], and attracted the interest of researchers in subsequent years on how to optimize the

system structure to achieve maximum system reliability and minimum system average cost. Some authors devoted to mathematically optimizing system structure using various constraints: Phillips [9] proved that the k -out-of- n system is preferable for maximizing the reliability of the system with i.i.d. components subject to two failure modes for all values of n ; Ben-Dov [1] derived the optimal k value that maximizes system reliability for a fixed n ; Sah and Stiglitz [10] derived several properties of k -out-of- n systems and determined the optimal value of k that maximizes the mean system profit for a given n ; Pham [5] derived the optimal n value that maximizes system reliability for a fixed k ; Pham and Pham [8] determined the optimal k and n values for maximizing system profit; Pham and Malon [7] determined the optimal k and n values for minimizing system cost.

The above authors did great work on the two-failure mode problem of single k -out-of- n systems; however, their results cannot be applied directly to a model that consists of two k -out-of- $n:F$ subsystems with multi-failure modes. An application can be found in underground coal mine monitoring systems.

3. Underground Coal Mine Monitoring Systems

In underground mining, work safety and productivity are correlated closely with environmental quality and equipment operating conditions. Due to the harsh working conditions in underground coal mines, monitoring and detection of mine fires and other environmental variables like the hazardous and explosive gases are significant concerns in mine production safety.

Consider a monitoring sensor group and controller subsystem shown in Figure 1. Assuming the outstation (controller), power supply, alarm box, and relay box have 100% reliability, the subsystem can be simplified to the general model, which consists of n_1 i.i.d. methane sensors, denoted as $S_1, S_2, \dots, S_{n1}$, and n_2 i.i.d. smoke sensors, denoted as $X_1, X_2, \dots, X_{n2}$. Methane sensors can fail in two ways: by reporting a reading over the threshold while the real methane content has not reached the power breaking point (mode 1), or by failing to detect methane content that has already reached the power breaking point (mode 2). Smoke sensors have similar failure modes (mode 3 and mode 4).

Since each failure mode can cause the loss of production or damage of equipment, problems regarding this kind of subsystem are important. The effect of redundancy on system reliability and average system cost are matters of concern to all involved. So this paper, we will discuss the optimal design for the model

of two k -out-of- $n:F$ subsystems connected in series with four failure modes.

Figure 1. A simplified monitoring sensor group and controller subsystem

4. Two k -out-of- n Subsystems Connected in Series

Since we have studied reliability problems in the underground coal mine monitoring system and built up a simplified model for its crucial subsystem, now, we will provide an optimal design for two k -out-of- $n:F$ subsystems connected in series with four failure modes, and determine:

- the optimal values of k_1^*, k_2^* that maximize system reliability and minimize average system cost, for fixed n_1, n_2 ;
- the optimal values of n_1^*, n_2^* that maximize system reliability and minimize average system cost, for fixed k_1, k_2 under the constraint $n_1 + n_2 \leq n$.

The following assumption and notation are used in this study:

Assumption:

- The subsystems consist of $n_1 + n_2$ sensors that are statistically independent, where n_1 and n_2 are two groups of sensors and identical respectively.
- Each sensor can be either good (accurate) or failed (error exceeds toleration or the sensor is completely unable to work).
- The system can be: good (it sends correct commands) or failed in mode 1, 2, 3, or 4, respectively.
- The failures of the two types of sensors are independent.

- The constraint is: $n_1 + n_2 \leq n$, where, n is the largest number of sensors (inputs) connected to the controller.
- The probabilities of sensor failure $q_{11}, q_{12}, q_{21}, q_{22}$ at mode 1, 2, 3 and 4 are known.
- Costs of system failure $c_{11}, c_{12}, c_{21}, c_{22}$ at mode 1, 2, 3 and 4 are known.
- The sensors are neither totally reliable nor totally unreliable, that is,
 $q_{11} > 0, q_{12} > 0, q_{21} > 0, q_{22} > 0$, and, $q_{11} + q_{12} < 1, q_{21} + q_{22} < 1$. Thus,
 $q_{11} + p_{11} = 1 \Rightarrow q_{12} < p_{11};$
 $q_{21} + p_{21} = 1 \Rightarrow q_{22} < p_{21};$
 $q_{12} + p_{12} = 1 \Rightarrow q_{11} < p_{12};$
 $q_{22} + p_{22} = 1 \Rightarrow q_{21} < p_{22}.$

Notation:

- q_{11} : probability of a sensor (group 1)'s failure in mode 1 ($p_{11} = 1 - q_{11}$)
- q_{12} : probability of a sensor (group 1)'s failure in mode 2 ($p_{12} = 1 - q_{12}$)
- q_{21} : probability of a sensor (group 2)'s failure in mode 3 ($p_{21} = 1 - q_{21}$)
- q_{22} : probability of a sensor (group 2)'s failure in mode 4 ($p_{22} = 1 - q_{22}$)
- F_{11} : probability that the system fails in mode 1
- F_{12} : probability that the system fails in mode 2
- F_{21} : probability that the system fails in mode 3
- F_{22} : probability that the system fails in mode 4
- F_1 : failure probability of the menthane sensor sub-system ($F_1 = F_{11} + F_{12}$)
- F_2 : failure probability of the smoke sensor subsystem ($F_2 = F_{21} + F_{22}$)
- R_1 : reliability of the menthane sensor subsystem ($R_1 = 1 - F_1$)
- R_2 : reliability of the smoke sensor subsystem ($R_2 = 1 - F_2$)
- F_s : system's failure probability
- R_s : system's reliability ($R_s = 1 - F_s$)
- c_{11} : loss when system fails in mode 1
- c_{12} : loss when system fails in mode 2
- c_{21} : loss when system fails in mode 3
- c_{22} : loss when system fails in mode 4

- d_1 : the cost of each sensor from group 1
- d_2 : the cost of each sensor from group 2
- $T(k_1, n_1, k_2, n_2)$: total system cost

4.1. Maximizing system reliability

Consider the simplified model illustrated in Figure 1. The system fails in mode 1 if and only if at least k_1 of its n_1 methane sensors from group 1 fail in mode 1. We obtain the probability that the system fails in mode 1, denoted as $F_{11}(k_1, n_1)$:

$$\begin{aligned} F_{11}(k_1, n_1) &= \sum_{i=k_1}^{n_1} \binom{n_1}{i} q_{11}^i p_{11}^{n_1-i} \\ &= 1 - \sum_{i=0}^{k_1-1} \binom{n_1}{i} q_{11}^i p_{11}^{n_1-i}. \end{aligned} \quad (5)$$

The system fails in mode 2 if and only if at least $n_1 - k_1 + 1$ of its n_1 methane sensors from group 1 fail in mode 2. We obtain the probability that the system fails in mode 2, denoted as $F_{12}(k_1, n_1)$:

$$\begin{aligned} F_{12}(k_1, n_1) &= \sum_{i=n_1-k_1+1}^{n_1} \binom{n_1}{i} q_{12}^i p_{12}^{n_1-i} \\ &= \sum_{i=0}^{k_1-1} \binom{n_1}{i} p_{12}^i q_{12}^{n_1-i}. \end{aligned} \quad (6)$$

It can also be shown that:

$$\begin{aligned} F_{21}(k_2, n_2) &= \sum_{i=k_2}^{n_2} \binom{n_2}{i} q_{21}^i p_{21}^{n_2-i} \\ &= 1 - \sum_{i=0}^{k_2-1} \binom{n_2}{i} q_{21}^i p_{21}^{n_2-i}, \end{aligned} \quad (7)$$

$$\begin{aligned} F_{22}(k_2, n_2) &= \sum_{i=n_2-k_2+1}^{n_2} \binom{n_2}{i} q_{22}^i p_{22}^{n_2-i} \\ &= \sum_{i=0}^{k_2-1} \binom{n_2}{i} p_{22}^i q_{22}^{n_2-i}. \end{aligned} \quad (8)$$

Then, the system's reliability is expressed as:

$$R_s = 1 - F_s = R_1 \times R_2. \quad (9)$$

This means that we have two k -out-of- n :F subsystems connected in series. Define,

$$B(k; p, n) = \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i}.$$

Then, the system reliability in equation (9) can be written as:

$$\begin{aligned}
R_S &= R_1 \times R_2 = [1 - F_1][1 - F_2] \\
&= [1 - F_{11} - F_{12}][1 - F_{21} - F_{22}] \\
&= \left[\sum_{i=0}^{k_1-1} \binom{n_1}{i} q_{11}^i p_{11}^{n_1-i} - \sum_{i=0}^{k_1-1} \binom{n_1}{i} p_{12}^i q_{12}^{n_1-i} \right] \\
&\times \left[\sum_{i=0}^{k_2-1} \binom{n_2}{i} q_{21}^i p_{21}^{n_2-i} - \sum_{i=0}^{k_2-1} \binom{n_2}{i} p_{22}^i q_{22}^{n_2-i} \right] \\
&= [B(k_1 - 1; q_{11}, n_1) - B(k_1 - 1; p_{12}, n_1)] \\
&\times [B(k_2 - 1; q_{21}, n_2) - B(k_2 - 1; p_{22}, n_2)]. \quad (10)
\end{aligned}$$

Theorem 1: Fixing $n_1, n_2, q_{11}, q_{12}, q_{21}, q_{22}$, where $n_1 + n_2 \leq n$, the optimal value of k_1, k_2 for maximizing system reliability, $R(k_1, k_2)$, say k_1^*, k_2^* can be obtained at,

$$\begin{cases} k_1^* = \lfloor k_{01} \rfloor + 1 \\ k_2^* = \lfloor k_{02} \rfloor + 1, \end{cases} \quad (11)$$

where

$$k_{01} = n_1 \frac{\ln\left(\frac{q_{12}}{p_{11}}\right)}{\ln\left(\frac{q_{11}q_{12}}{p_{11}p_{12}}\right)}, \quad (12)$$

$$k_{02} = n_2 \frac{\ln\left(\frac{q_{22}}{p_{21}}\right)}{\ln\left(\frac{q_{21}q_{22}}{p_{21}p_{22}}\right)}. \quad (13)$$

The proof of Theorem 1 is given in Duan [3].

Example 1 Given $n_1 = 14, n_2 = 6, q_{11} = 0.03, q_{12} = 0.08, q_{21} = 0.06, q_{22} = 0.04$, by equations (12) and (13), we obtain $k_{01} = 5.9025, k_{02} = 3.1945$, then, $k_1^* = 6, k_2^* = 4$. Substituting k_1^*, k_2^* into equation (10), we see that $R_{max} = 0.9987$. To test the accuracy of this optimal value, we can use Matlab to write a simple recursive algorithm and the maximum reliability $R_{max} = 0.9987$ at $k_1^* = 6, k_2^* = 4$, which are the same as the results given from Theorem 1.

From this example we can find that though the failure probability of methane sensors and smoke sensors at each mode are not very low, the system reliability can still reach 0.9987 by setting optimal k_1, k_2 value, which proves that setting optimal redundancy can greatly improve the system reliability in two k -out-of- n :F subsystems connected in series.

Theorem 2: Fixing $k_1, k_2, q_{11}, q_{12}, q_{21}, q_{22}$, the

optimal values of n_1, n_2 that maximize the system reliability, $R(n_1, n_2)$, say n_1^*, n_2^* can be obtained at,

$$\begin{cases} n_1^* = \lfloor n_{01} \rfloor + 1 \\ n_2^* = \lfloor n_{02} \rfloor + 1, \end{cases} \quad (14)$$

where

$$n_{01} = k_1 \left[1 + \frac{\ln\left(\frac{p_{12}}{q_{11}}\right)}{\ln\left(\frac{p_{11}}{q_{12}}\right)} \right] - 1, \quad (15)$$

$$n_{02} = k_2 \left[1 + \frac{\ln\left(\frac{p_{22}}{q_{21}}\right)}{\ln\left(\frac{p_{21}}{q_{22}}\right)} \right] - 1. \quad (16)$$

The proof of Theorem 2 is given in Duan [3].

Example 2 Given $k_1 = 7, k_2 = 8, q_{11} = 0.17, q_{12} = 0.15, q_{21} = 0.10, q_{22} = 0.20$, by equations (15) and (16), we obtain $n_{01} = 13.585, n_{02} = 19.060$, then, $n_1^* = 14, n_2^* = 20$. Substituting k_1^*, k_2^* into equation (10), we see that $R_{max} = 0.9957$. Again, by applying the simple recursive algorithm, we can get the same results.

A problem is that the optimal value n_1^* and n_2^* given by equation (15) and (16) are derived without considering the constraint $n_1 + n_2 \leq n$. In practice, we can also compute such n_1^*, n_2^* , then see whether they satisfy the constraint or not. If $n_1^* + n_2^* > n$, either n_1^* or n_2^* or both must be decreased until they have satisfied the constraint.

4.2. Minimizing the average system cost

From last section, we can conclude that as the reliability of two k -out-of- n :F subsystems connected in series increases, the average system cost increases as a result of using more redundant sensors. In practise, designers must consider the economic issues as well as the improvement of reliability. Thus, our concern is how to derive a cost-effective solution that achieves a balance between reliability and cost.

To simplify the problem, we assume that the failures of two types of sensors are independent and mutually exclusive, that is:

$$F_s = F_1 + F_2 = F_{11} + F_{12} + F_{21} + F_{22}.$$

This assumption makes sense when F_1 and F_2 are small. Define the total system cost $T(k_1, n_1, k_2, n_2)$:

$$\begin{aligned}
T(k_1, n_1, k_2, n_2) &= T(k_1, n_1) + T(k_2, n_2) \\
&= d_1 n_1 + c_{11} F_{11}(k_1, n_1) + c_{12} F_{12}(k_1, n_1) \\
&+ d_2 n_2 + c_{21} F_{21}(k_2, n_2) + c_{22} F_{22}(k_2, n_2). \quad (17)
\end{aligned}$$

The objective function (17) is the cost of all sensors in the subsystem, plus the expected cost of subsystem failure in mode 1 ($c_{11}F_{11}(k_1, n_1)$), mode 2 ($c_{12}F_{12}(k_1, n_1)$), mode 3 ($c_{21}F_{21}(k_2, n_2)$), and mode 4 ($c_{22}F_{22}(k_2, n_2)$). We also have the constraint of $n_1 + n_2 \leq n$.

Theorem 3: Fixing $n_1, n_2, q_{11}, q_{12}, q_{21}, q_{22}, d_1, d_2, c_{11}, c_{12}, c_{21}$ and c_{22} , where $n_1 + n_2 \leq n$, the optimal values of k_1, k_2 that minimize total system cost, $T(k_1, k_2)$, say k_1^*, k_2^* , can be obtained at

$$\begin{cases} k_1^* = \min\{n_1, \max\{1, \lfloor k_{01} \rfloor + 1\}\} \\ k_2^* = \min\{n_2, \max\{1, \lfloor k_{02} \rfloor + 1\}\}, \end{cases} \quad (18)$$

where

$$k_{01} = \frac{\ln\left(\frac{c_{11}}{c_{12}}\right) + n_1 \ln\left(\frac{p_{11}}{q_{12}}\right)}{\ln\left(\frac{p_{11}p_{12}}{q_{11}q_{12}}\right)}, \quad (19)$$

$$k_{02} = \frac{\ln\left(\frac{c_{21}}{c_{22}}\right) + n_2 \ln\left(\frac{p_{21}}{q_{22}}\right)}{\ln\left(\frac{p_{21}p_{22}}{q_{21}q_{22}}\right)}. \quad (20)$$

The proof of Theorem 3 is given in Duan [3].

Example 3 Given $n_1 = 11, n_2 = 15, c_{11} = 150000, c_{12} = 10000, c_{21} = 80000, c_{22} = 5000, d_1 = 1000, d_2 = 500, q_{11} = 0.11, q_{12} = 0.02, q_{21} = 0.10, q_{22} = 0.01$, by equations (18) to (20), we obtain $k_{01} = 7.4313, k_{02} = 10.3454$, then, $k_1^* = 8, k_2^* = 11$. Substituting k_1^*, k_2^* into equation (17), we see that $T_{\min} = 18501$. Using Matlab to write a simple recursive algorithm, we obtain the same minimum system cost $T_{\min} = 18501$ at $k_1^* = 8, k_2^* = 11$, which show the accuracy of Theorem 3.

Then we try to obtain n_1^*, n_2^* as k_1, k_2 are given. For two k -out-of- n :F subsystems connected in series with four failure modes, when k_1, k_2 are given, there are two variables, n_1, n_2 , in the total system cost equation (17). If the constraint of $n_1 + n_2 \leq n$ is considered, n_1 and n_2 are dependent; if the constraint is ignored, n_1 and n_2 are independent.

To simplify the problem, first, we ignore the constraint, $n_1 + n_2 \leq n$, so n_1 and n_2 are independent in the objective function (17).

Theorem 4: Fixing $k_1, k_2, q_{11}, q_{12}, q_{21}, q_{22}, d_1, d_2, c_{11}, c_{12}, c_{21}$ and c_{22} , the optimal values of n_1, n_2 that minimize the average system cost, $T(n_1, n_2)$, say n_1^*, n_2^* , can be seen to be:

- If $n_{01} \leq k_1, n_{02} \leq k_2, n_1^* = k_1, n_2^* = k_2$, where

$$n_{01} = \left\lceil \frac{\ln\left(\frac{c_{12}}{c_{11}}\right) + k_1 \ln\left(\frac{p_{11}p_{12}}{q_{11}q_{12}}\right)}{\ln\left(\frac{p_{11}}{q_{12}}\right)} - 1 \right\rceil,$$

$$n_{02} = \left\lceil \frac{\ln\left(\frac{c_{22}}{c_{21}}\right) + k_2 \ln\left(\frac{p_{21}p_{22}}{q_{21}q_{22}}\right)}{\ln\left(\frac{p_{21}}{q_{22}}\right)} - 1 \right\rceil.$$

- If $n_{01} > k_1$,

1. if $h(n_{h1}) < \frac{d_1}{c_{11}}$, then $n_1^* = k_1$;
2. if $h(n_{h1}) \geq \frac{d_1}{c_{11}}$, and $h(k_1) \geq \frac{d_1}{c_{11}}$, then $n_1^* = n_{m1}$;
3. if $h(n_{h1}) \geq \frac{d_1}{c_{11}}$, and $h(k_1) < \frac{d_1}{c_{11}}$, then,

$$n_1^* = \begin{cases} k_1, & \text{if } T(k_1, k_1) \leq T(k_1, n_{m1}) \\ n_{m1}, & \text{if } T(k_1, k_1) > T(k_1, n_{m1}), \end{cases} \quad (21)$$

where

$$h(n_1) = \left(\frac{n_1}{k_1 - 1}\right) p_{12}^{k_1} q_{12}^{n_1 - k_1 + 1} \times \left[\frac{c_{12}}{c_{11}} - \left(\frac{q_{11}q_{12}}{p_{11}p_{12}}\right)^{k_1} \left(\frac{p_{11}}{q_{12}}\right)^{n_1 + 1} \right], \quad (22)$$

$$n_{h1} = f^{-1}(B_1), \quad \text{for } k_1 \leq n_{h1} \leq n_{t1},$$

$$n_{m1} = \inf \left\{ n_1 \in [n_{h1}, n_{01}] : h(n_1) < \frac{d_1}{c_{11}} \right\}.$$

Define

$$B_1 = \frac{c_{12}}{c_{11}} \left(\frac{p_{11}p_{12}}{q_{11}q_{12}}\right)^{k_1} \left(\frac{q_{12}}{p_{11}}\right),$$

$$f(n_1) = \left(\frac{p_{11}}{q_{12}}\right)^{n_1} \frac{(n_1 + 1)q_{11} - (k_1 - 1)}{(n_1 + 1)p_{12} - (k_1 - 1)},$$

$$n_{t1} = \left\lceil \frac{k_1 - 1}{1 - q_{12}} - 1 \right\rceil.$$

- If $n_{02} > k_2$,

1. if $h(n_{h2}) < \frac{d_2}{c_{21}}$, then $n_2^* = k_2$;
2. if $h(n_{h2}) \geq \frac{d_2}{c_{21}}$, and $h(k_2) \geq \frac{d_2}{c_{21}}$, then $n_2^* = n_{m2}$;
3. if $h(n_{h2}) \geq \frac{d_2}{c_{21}}$, and $h(k_2) < \frac{d_2}{c_{21}}$, then,

$$n_2^* = \begin{cases} k_2, & \text{if } T(k_2, k_2) \leq T(k_2, n_{m2}) \\ n_{m2}, & \text{if } T(k_2, k_2) > T(k_2, n_{m2}), \end{cases} \quad (23)$$

where

$$h(n_2) = \binom{n_2}{k_2-1} p_{22}^{k_2} q_{22}^{n_2-k_2+1} \times \left[\frac{c_{22}}{c_{21}} - \left(\frac{q_{21}q_{22}}{p_{21}p_{22}} \right)^{k_2} \left(\frac{p_{21}}{q_{22}} \right)^{n_2+1} \right], \quad (24)$$

$$n_{h2} = f^{-1}(B_2), \quad \text{for } k \leq n_{h2} \leq n_{t2},$$

$$n_{m2} = \inf \left\{ n_2 \in [n_{h2}, n_{02}] : h(n_2) < \frac{d_2}{c_{21}} \right\}.$$

Define

$$B_2 = \frac{c_{22}}{c_{21}} \left(\frac{p_{21}p_{22}}{q_{21}q_{22}} \right)^{k_2} \left(\frac{q_{22}}{p_{21}} \right),$$

$$f(n_2) = \left(\frac{p_{21}}{q_{22}} \right)^{n_2} \frac{(n_2+1)q_{21} - (k_2-1)}{(n_2+1)p_{22} - (k_2-1)},$$

$$n_{t2} = \lceil \frac{k_2-1}{1-q_{22}} - 1 \rceil.$$

The proof of Theorem 4 is given in Duan [3].

Example 4 Given $k_1 = 5$, $k_2 = 9$, $c_{11} = 10000$, $c_{12} = 10000$, $c_{21} = 6000$, $c_{22} = 150000$, $d_1 = 1000$, $d_2 = 600$, $q_{11} = 0.04$, $q_{12} = 0.02$, $q_{21} = 0.03$, $q_{22} = 0.01$, by applying the simple recursive algorithm, we obtain the minimum average system cost $T_{min} = 12584.1$, $n_1^* = 5$, and $n_2^* = 11$.

Now let us see if we can get the same result by applying Theorem 4. For instance, substituting the initial data for case 1 into equations (21) and (23),

- we have $n_{01} = 8 > k_1 = 5$, $n_{h1} \simeq 5$, and $h(n_{h1}) = 0.0904 < \frac{d_1}{c_{11}} = 0.1$. This satisfies the second condition in Theorem 4; that is, when $n_{01} > k_1$, if $h(n_{h1}) < \frac{d_1}{c_{11}}$, then, $n_1^* = k_1$. Thus we obtain the optimal value: $n_1^* = k_1 = 5$.
- we have $n_{02} = 15 > k_2 = 9$, $n_{h2} \simeq 9$, $h(n_{h2}) = h(k_2) = 2.0554 > \frac{d_2}{c_{21}} = 0.1$, $n_{m2} = 11$, and $h(11) = 0.0226 < \frac{d_2}{c_{21}} = 0.1$. This satisfies the third condition in Theorem 4; that is, when $n_{02} > k_2$, if $h(n_{h2}) \geq \frac{d_2}{c_{21}}$ and $h(k_2) \geq \frac{d_2}{c_{21}}$ then, $n_2^* = n_{m2}$. Thus we obtain the optimal value: $n_2^* = n_{m2} = 11$.

So by applying Theorem 4 we obtain the same optimal values for n_1^*, n_2^* . Subsequently, we'll examine whether or not $n_1 + n_2 \leq n$.

As illustrated in (21) and (23), n_1^*, n_2^* have multi solutions at different values of n_{01} , n_{02} , $h(n_1)$ and $h(n_2)$.

And the following solutions for n_1^*, n_2^* satisfy the constraint.

Case 1: when $n_{01} \leq k_1$, $n_{02} \leq k_2$,

$$\begin{cases} n_1^* = k_1 \\ n_2^* = k_2 \end{cases}$$

Since $k_1 + k_2 \leq n$ is an initial condition which has been given, obviously, $n_1^* + n_2^* \leq n$, $n_1^* = k_1$, $n_2^* = k_2$ is a solution which minimizes the objective function (the expected system cost) under the constraint $n_1 + n_2 \leq n$. We can obtain the same solution when n_{01} , n_{02} , $h(n_1)$ and $h(n_2)$ are at different values in case 2 and case 3 as below.

Case 2: when $n_{01} > k_1$, $h(n_{h1}) < \frac{d_1}{c_{11}}$, $n_{02} > k_2$, $h(n_{h2}) < \frac{d_2}{c_{21}}$,

$$\begin{cases} n_1^* = k_1 \\ n_2^* = k_2 \end{cases}$$

Case 3: when $n_{01} > k_1$, $h(n_{h1}) \geq \frac{d_1}{c_{11}}$, $h(k_1) < \frac{d_1}{c_{11}}$, $T(k_1, k_1) \leq T(k_1, n_{m1})$; $n_{02} > k_2$, $h(n_{h2}) \geq \frac{d_2}{c_{21}}$, $h(k_2) < \frac{d_2}{c_{21}}$, $T(k_2, k_2) \leq T(k_2, n_{m2})$;

$$\begin{cases} n_1^* = k_1 \\ n_2^* = k_2 \end{cases}$$

Others:

$$\begin{cases} n_1^* = k_1 \\ n_2^* = n_{m2} \end{cases} \quad \begin{cases} n_1^* = n_{m1} \\ n_2^* = k_2 \end{cases} \quad \begin{cases} n_1^* = n_{m1} \\ n_2^* = n_{m2} \end{cases}$$

Since $n_{m1} \geq n_{h1} \geq k_1$, $n_{m2} \geq n_{h2} \geq k_2$, it's possible that, $n_1^* + n_2^* = k_1 + n_{m2} > n$, or $n_1^* + n_2^* = k_2 + n_{m1} > n$, or $n_1^* + n_2^* = n_{m1} + n_{m2} > n$. Thus, n_1^* and n_2^* in these cases can be the optimal solution if, and only if $n_1^* + n_2^* \leq n$.

4.3. Find n_1^*, n_2^* , if $n_1^* + n_2^* \leq n$

In the above section, we found the optimal values of n_1^*, n_2^* that satisfied the constraint of $n_1^* + n_2^* \leq n$. If $n_1^* + n_2^* > n$, we must find new n_1^* and n_2^* so that $n_1^* + n_2^* = n$.

There are two approaches we can use to solve the above problem.

Approach 1:

If the current solution violate the constraint $n_1^* + n_2^* > n$, either n_1^* or n_2^* or both must be decreased.

First check the following quantities:

$$\Delta_1 \equiv T_1(n_1^* - 1) - T_1(n_1^*) \geq 0,$$

$$\Delta_2 \equiv T_2(n_2^* - 1) - T_1(n_2^*) \geq 0.$$

Both Δ quantities must be greater than or equal to 0. Then choose the smaller Δ value for changing the corresponding n_1^* or n_2^* value, and obtain a new pair of n_1^* and n_2^* values. Thus, we obtain new values of the Δ 's.

Repeat this process until $n_1^* + n_2^* = n$.

Approach 2:

Substituting $n_2 = n - n_1$ for n_2 into the objective function, the objective function will contain only one variable, n_1 , as below,

$$\begin{aligned} T(k_1, n_1, k_2, n_2) &= T(k_1, n_1) + T(k_2, n - n_1) \\ &= d_1 n_1 + c_{11} F_{11}(k_1, n_1) + c_{12} F_{12}(k_1, n_1) \\ &+ d_2(n - n_1) + c_{21} F_{21}(k_2, n - n_1) \\ &+ c_{22} F_{22}(k_2, n - n_1). \end{aligned} \quad (25)$$

Define

$$\Delta T(k_1, k_2, n_1) = \Delta T(n_1) + \Delta T(n - n_1). \quad (26)$$

where

$$\Delta T(n_1) = T(n_1 + 1) - T(n_1),$$

$$\Delta T(n - n_1) = T(n - n_1 - 1) - T(n - n_1).$$

Then

$$\begin{aligned} \Delta T(n_1) &= T(n_1 + 1) - T(n_1) \\ &= d_1 + c_{11} \binom{n_1}{k_1 - 1} q_{11}^{k_1} p_{11}^{n_1 - k_1 + 1} \\ &- c_{12} \binom{n_1}{k_1 - 1} p_{12}^{k_1} q_{12}^{n_1 - k_1 + 1} \\ &= d_1 - c_{11} \binom{n_1}{k_1 - 1} p_{12}^{k_1} q_{12}^{n_1 - k_1 + 1} \\ &\times \left[\frac{c_{12}}{c_{11}} - \left(\frac{q_{11} q_{12}}{p_{11} p_{12}} \right)^{k_1} \left(\frac{p_{11}}{q_{12}} \right)^{n_1 + 1} \right] \\ &= d_1 - c_{11} h(n_1), \end{aligned} \quad (27)$$

$$\begin{aligned} \Delta T(n - n_1) &= T(n - n_1 - 1) - T(n - n_1) \\ &= -d_2 + c_{21} \binom{n - n_1 - 1}{k_2 - 1} p_{22}^{k_2} q_{22}^{n - n_1 - k_2} \\ &\times \left[\frac{c_{22}}{c_{21}} - \left(\frac{q_{21} q_{22}}{p_{21} p_{22}} \right)^{k_2} \left(\frac{p_{21}}{q_{22}} \right)^{n - n_1} \right] \\ &= -d_2 + c_{21} g(n_1), \end{aligned} \quad (28)$$

where

$$\begin{aligned} g(n_1) &= \binom{n - n_1 - 1}{k_2 - 1} p_{22}^{k_2} q_{22}^{n - n_1 - k_2} \\ &\times \left[\frac{c_{22}}{c_{21}} - \left(\frac{q_{21} q_{22}}{p_{21} p_{22}} \right)^{k_2} \left(\frac{p_{21}}{q_{22}} \right)^{n - n_1} \right]. \end{aligned} \quad (29)$$

From equation (28), we can see that $g(n_1) \leq 0$, $\Delta T(n - n_1) \leq 0$, which implies that group 1's size of $n - n_1 - 1$ is better than the size of $n - n_1$. So we define

$$n_{03} = \left\lfloor n - \frac{\ln \left(\frac{c_{22}}{c_{21}} \right) + k_2 \ln \left(\frac{p_{21} p_{22}}{q_{21} q_{22}} \right)}{\ln \left(\frac{p_{21}}{q_{22}} \right)} \right\rfloor.$$

Then, we can easily get $g(n_1) \leq 0$, for $k_2 \leq n_1 \leq n_{03}$.

Let

$$\Delta g(n_1) = g(n_1 + 1) - g(n_1), \quad (30)$$

we get

$$\begin{aligned} \Delta g(n_1) &= \frac{\binom{n - n_1 - 1}{k_2 - 1} p_{22}^{k_2} q_{22}^{n - n_1 - k_2}}{(n - n_1 - k_2 - 1)} \left\{ \frac{c_{22}}{c_{21}} [(n - n_1 \right. \\ &- 1) p_{22} - k_2] - \left(\frac{q_{21} q_{22}}{p_{21} p_{22}} \right)^{k_2} \left(\frac{p_{21}}{q_{22}} \right)^{n - n_1} \\ &\times [(n - n_1 - 1) q_{21} - k_2] \}. \end{aligned} \quad (31)$$

When $\Delta g(n_1) = 0$, we have

$$\begin{aligned} &\frac{c_{22}}{c_{21}} \left(\frac{p_{21} p_{22}}{q_{21} q_{22}} \right)^{k_2} \left(\frac{q_{22}}{p_{21}} \right)^n \\ &= \left(\frac{q_{22}}{p_{21}} \right)^{n_1} \frac{(n - n_1 - 1) q_{21} - k_2}{(n - n_1 - 1) p_{22} - k_2}. \end{aligned} \quad (32)$$

Define

$$\begin{aligned} A &= \frac{c_{22}}{c_{21}} \left(\frac{p_{21} p_{22}}{q_{21} q_{22}} \right)^{k_2} \left(\frac{q_{22}}{p_{21}} \right)^n, \\ l(n_1) &= \left(\frac{q_{22}}{p_{21}} \right)^{n_1} \frac{(n - n_1 - 1) q_{21} - k_2}{(n - n_1 - 1) p_{22} - k_2}, \\ n'_t &= \left\lceil n - \frac{k_2}{1 - q_{22}} - 1 \right\rceil, \end{aligned}$$

$$n'_h = l^{-1}(A), \quad \text{for } k_2 \leq n'_h \leq n'_t.$$

We can prove that the function $g(n_1)$ decreases in n_1 for $n_1 \in [k_2, n'_h]$ and increases in n_1 for $n_1 \in [n'_h, n_{03}]$. This result shows that $g(n_1)$ is unimodal and achieves a minimum value at $n_1 = n'_h$ for $k_2 \leq n_1 \leq n_{03}$; also, in this interval, $g(n_1) \leq 0$.

Substituting (27) and (28) into equation (26), we have

$$\Delta T(k_1, k_2, n_1) = d_1 - c_{11} h(n_1) - d_2 + c_{21} g(n_1). \quad (33)$$

Since, $h(n_1)$ is unimodal and has a maximum value at $n_1 = n_{h1}$ for $k_1 \leq n_1 \leq n_{01}$, and $g(n_1)$ is unimodal and has a minimum value at $n_1 = n'_h$ for $k_2 \leq n_1 \leq n_{03}$, there must exist an optimal n_1^* which lets $\Delta T(k_1, k_2, n_1^*) = 0$, and allows the expected system cost, $T(k_1, n_1, k_2, n - n_2)$, to be minimized. As well, using $n_2^* = n - n_1^*$, we can determine n_2^* .

5. Conclusion

This paper reports an application of the two k -out-of- n :F subsystems model in underground coal mine monitoring systems. Current mine monitoring system designs usually focus on system functions and intrinsically safe circuits, but ignore the system reliability and structure optimization. This research provides the optimal design to obtain a balance between a more reliable system and most economical number of redundancy in each subsystem. Under this case, the optimal methods and procedures can also be used in similar structure but different applications. It should be noted that there are no reported research results for the problem of the two k -out-of- n :F subsystems connected in series in the literature.

This article determines the optimal values of k_1^* , k_2^* that maximize the system reliability and minimize the average system cost for fixed n_1 , n_2 ; and the optimal values of n_1^* , n_2^* that maximize the system reliability and minimize the average system cost for fixed k_1 , k_2 . While these problems have been studied in great detail, one area which has not been sufficiently analyzed is to obtain the optimal values of n_1^* , n_2^* under the constraint $n_1 + n_2 \leq n$. In practice, most industrial systems have the limitation of the total component number. This problem is crucial for development of realistic models. Two approaches to decrease n_1 , n_2 as the constraint is violated are recommended in the end of section 4, which will be helpful to derive the mathematical solution in the future research.

Along the line of this paper, future studies could be done in the following areas:

- Deriving a mathematical solution for the situation in which the constraint of $n_1 + n_2 \leq n$ is violated;
- Determining the optimal values (k_1^* , k_2^* , n_1^* , n_2^*) for maximizing system reliability and minimizing average system cost, under the constraint, $n_1 + n_2 \leq n$;

- Changing particular assumptions in order to make the model more typical.

Acknowledgement

This research is partially supported by the Natural Sciences and Engineering Research Council of Canada.

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